

Tracefree $\mathrm{SL}(2, \mathbb{C})$ -representations of Montesinos links

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Abstract

Given a link L , a representation $\pi_1(S^3 - L) \rightarrow \mathrm{SL}(2, \mathbb{C})$ is *tracefree* if the image of each meridian has trace zero. We determine the conjugacy classes of tracefree representations when L is a Montesinos link.

1 Introduction

Given a link $L \subset S^3$ and a linear group G , a *tracefree* (or *traceless*) G -representation of L means a homomorphism $\pi_1(S^3 - L) \rightarrow G$ sending each meridian to an element of trace zero. Dated back to 1980, Magnus [5] used tracefree $\mathrm{SL}(2, \mathbb{C})$ -representations to prove the faithfulness of a representation of braid groups in the automorphism groups of the rings generated by the characters functions on free groups. Lin [4] used tracefree $\mathrm{SU}(2)$ -representations to define a Casson-type invariant of a knot K , and showed it to equal half of the signature of K . More interestingly, Kronheimer and Mrowka [3] observed that for some knots K , its Khovanov homology is isomorphic to the ordinary homology of the space $R(K)$ of conjugacy classes of tracefree representations of K . In this context, Zentner [8] determined $R(K)$ when K belongs to a class of classical pretzel knots. For related works, one may refer to [1], [2], etc.

There are relatively fewer results on tracefree $\mathrm{SL}(2, \mathbb{C})$ -representations. For a knot k , Nagasato [6] gave a set of polynomials whose zero loci is exactly the tracefree characters of irreducible $\mathrm{SL}(2, \mathbb{C})$ -representations of K . Nagasato and Yamaguchi [7] investigated tracefree $\mathrm{SL}(2, \mathbb{C})$ -representations of $\pi_1(\Sigma - K)$ (where K is a knot in an integral 3-sphere Σ), and related to those of $\pi_1(B_2)$, where B_2 is the 2-fold cover of Σ branched along K .

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In this paper, for each Montesinos link, we completely determine the tracefree $\mathrm{SL}(2, \mathbb{C})$ -representations by given explicit formulas.

Let $\mathrm{SL}_0(2, \mathbb{C}) = \{X \in \mathrm{SL}(2, \mathbb{C}) : \mathrm{tr}(X) = 0\}$. Note that each $X \in \mathrm{SL}_0(2, \mathbb{C})$ satisfies $X^{-1} = -X$.

By a “tangle” we mean an unoriented tangle diagram. Given a tangle T , let $\mathrm{Dar}(T)$ denote the set of directed arcs of T , (each arc gives two directed arcs). By a (*tracefree*) *representation* of a tangle diagram T , we mean a map $\rho : \mathrm{Dar}(T) \rightarrow \mathrm{SL}_0(2, \mathbb{C})$ such that $\rho(x^{-1}) = \rho(x)^{-1}$ for each $x \in \mathrm{Dar}(T)$ and at each crossing illustrated in Figure 1, $\rho(z) = \rho(x)\rho(y)\rho(x)^{-1}$. To present such a representation, it is sufficient to give each arc a direction and label an element of $\mathrm{SL}_0(2, \mathbb{C})$ beside it.

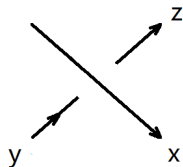


Figure 1: A representation satisfies $\rho(z) = \rho(x)\rho(y)\rho(x)^{-1}$ at each crossing

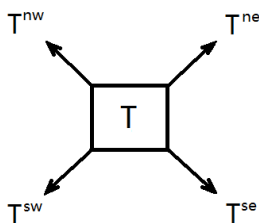


Figure 2: A tangle $T \in \mathcal{T}_2^2$, with the four ends directed outwards

Let \mathcal{T}_2^2 denote the set of tangles T with four ends which, when directed outwards, are denoted by $T^{nw}, T^{sw}, T^{ne}, T^{se}$, as shown in Figure 2. The simplest four ones are given in Figure 3. In \mathcal{T}_2^2 there are two binary operations: *horizontal composition* \cdot and *vertical composition* \star ; see Figure 4.

For $k \neq 0$, the horizontal composite of $|k|$ copies of $[1]$ (resp. $[-1]$) is denoted by $[k]$ if $k > 0$ (resp. $k < 0$), and the vertical composite of $|k|$ copies of $[1]$ (resp. $[-1]$) is denoted by $[1/k]$ if $k > 0$ (resp. $k < 0$).



Figure 3: The simplest four tangles: (a) $[0]$, (b) $[\infty]$, (c) $[1]$, (d) $[-1]$

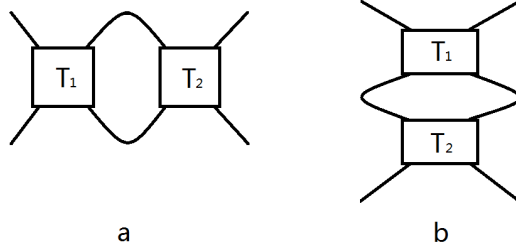


Figure 4: (a) $T_1 \cdot T_2$; (b) $T_1 \star T_2$

Given integers k_1, \dots, k_m , the *rational tangle* $[[k_1], \dots, [k_m]]$ is defined as

$$[k_1] \star [1/k_2] \cdot \dots \star [1/k_m], \quad \text{if } 2 \mid m, \quad (1)$$

$$[k_1] \star [1/k_2] \cdot \dots \cdot [k_m], \quad \text{if } 2 \nmid m; \quad (2)$$

and its *fraction* is defined by

$$f([[k_1], \dots, [k_m]]) = [[k_1, \dots, k_m]]^{(-1)^{m-1}}, \quad (3)$$

where the *continued fraction* $[[k_1, \dots, k_m]] \in \mathbb{Q}$ is defined inductively as

$$[[k_1]] = k_1, \quad [[k_1, \dots, k_m]] = k_m + 1/[[k_1, \dots, k_{m-1}]]. \quad (4)$$

Denote $[[k_1], \dots, [k_m]]$ as $[p/q]$ if the right-hand side of (3) equals p/q .

A *Montesinos tangle* is a tangle of the form $T = [p_1/q_1] \star \dots \star [p_r/q_r]$. The link obtained by connecting T^{nw} and T^{ne} with T^{sw} and T^{se} , respectively, is called a *Montesinos link* and denoted by $M(p_1/q_1, \dots, p_r/q_r)$.

Given a representation ρ of $T \in \mathcal{T}_{\text{ar}}$, denote

$$\rho^{\text{nw}} = \rho(T^{\text{nw}}), \quad \rho^{\text{sw}} = \rho(T^{\text{sw}}), \quad \rho^{\text{ne}} = \rho(T^{\text{ne}}), \quad \rho^{\text{se}} = \rho(T^{\text{se}}), \quad (5)$$

$$\text{tr}_v(\rho) = \text{tr}(\rho^{\text{ne}} \rho^{\text{se}}), \quad \text{tr}_h(\rho) = \text{tr}(\rho^{\text{sw}} \rho^{\text{se}}). \quad (6)$$

2 Representations of rational tangles

Suppose $[[k_1, \dots, k_m]] = [p/q]$ with $p, q \neq 0$. For a representation ρ of $[p/q]$, let X, Y, X_j, Y_j denote the elements that ρ assigns to the directed arcs shown in Figure 5. Call (X, Y) the *generating pair* of ρ , indicating that ρ is determined by X and Y .

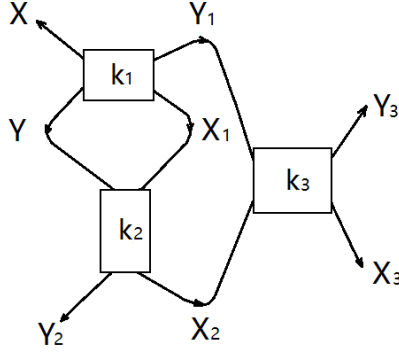


Figure 5: A representation of the rational tangle $[[k_1], [k_2], [k_3]]$

Suppose $\text{tr}(XY) = t$. Then $XYX^{-1} = -tX - Y$, hence each X_j or Y_j can be expressed as $\alpha(t)X + \beta(t)Y$, where $\alpha(t), \beta(t)$ are polynomials in t that do not depend on X, Y . We take a clever approach to derive formulas for the coefficients $\alpha(t), \beta(t)$.

For $a \in \mathbb{C}^\times$, put

$$A(a) = \frac{1}{2} \begin{pmatrix} (a + a^{-1})i & a - a^{-1} \\ a - a^{-1} & -(a + a^{-1})i \end{pmatrix}. \quad (7)$$

Lemma 2.1. (i) If ρ is a representation of $[k]$ with $\rho^{\text{nw}} = A(a_1)$ and $\rho^{\text{sw}} = A(a_2)$, then $\rho^{\text{ne}} = A(-a_1^{k+1}/a_2^k)$ and $\rho^{\text{se}} = A(-a_1^k/a_2^{k-1})$.

(ii) If ρ is a representation of $[1/k]$ with $\rho^{\text{nw}} = A(b_1)$ and $\rho^{\text{ne}} = A(b_2)$, then $\rho^{\text{sw}} = A(-b_1^{k+1}/b_2^k)$ and $\rho^{\text{se}} = A(-b_1^k/b_2^{k-1})$.

Proof. (i) Suppose ρ is a representation of $[1]$ with $\rho^{\text{nw}} = A(a_1)$ and $\rho^{\text{sw}} = A(a_2)$, then $\rho^{\text{se}} = A(-a_1)$, and computing directly,

$$\rho^{\text{ne}} = \rho^{\text{se}}(-\rho^{\text{sw}})(\rho^{\text{se}})^{-1} = A(-a_1)A(-a_2)A(-a_1)^{-1} = A(-a_1^2/a_2).$$

Applying this repeatedly, we obtain the result.

(ii) The proof is similar. □

Suppose $X = A(1)$ and $Y = A(s)$. By Lemma 2.1, $Y_1 = A(-s^{-k_1})$, $X_1 = A(-s^{1-k_1})$, and in general, $Y_j = A(s_j)$, $X_j = A(s'_j)$ with

$$s_0 = s, \quad s_1 = -s^{-k_1}, \quad s'_1 = -s^{1-k_1},$$

$$\frac{s_j}{s_{j-2}} = \frac{s'_j}{s'_{j-1}} = \left(\frac{s_{j-2}}{s'_{j-1}} \right)^{k_j}, \quad j \geq 2.$$

Consequently, for $j \geq 2$,

$$s'_j = \frac{s_j s_{j-1}}{s_1 s_0} s'_1 = s_j s_{j-1}, \quad s_j = (s_{j-1})^{-k_j} s_{j-2}.$$

Define u_j, v_j , $0 \leq j \leq m$, inductively by

$$\begin{aligned} u_0 &= 0, & u_1 &= 1, & u_{j+1} &= k_{j+1}u_j + u_{j-1}, & (j \geq 1), \\ v_0 &= 1, & v_1 &= k_1, & v_{j+1} &= k_{j+1}v_j + v_{j-1}, & (j \geq 1), \end{aligned}$$

so that $u_j/u_{j-1} = [[k_2, \dots, k_j]]$, $v_j/v_{j-1} = [[k_1, \dots, k_j]]$, then

$$Y_j = A\left((-1)^{u_j} s^{(-1)^j v_j}\right), \quad X_j = A\left((-1)^{u_j+u_{j-1}} s^{(-1)^j (v_j-v_{j-1})}\right). \quad (8)$$

$$\text{Put } \begin{cases} \tilde{p} = u_m, & \tilde{q} = u_{m-1}, & \text{if } 2 \nmid m, \\ \tilde{q} = u_m, & \tilde{p} = u_{m-1}, & \text{if } 2 \mid m, \end{cases} \quad \text{i.e.,}$$

$$\tilde{p}/\tilde{q} = [[k_2, \dots, k_m]]^{(-1)^{m-1}}. \quad (9)$$

One can prove by induction on m that

$$\tilde{p}q - p\tilde{q} = 1. \quad (10)$$

We have

$$\rho^{\text{ne}} = A((-1)^{\tilde{p}} s^{-p}), \quad \rho^{\text{sw}} = A((-1)^{\tilde{q}} s^q), \quad \rho^{\text{se}} = A((-1)^{\tilde{p}+\tilde{q}} s^{q-p}), \quad (11)$$

hence

$$-\text{tr}_v(\rho) = (-1)^{\tilde{q}}(s^q + s^{-q}), \quad -\text{tr}_h(\rho) = (-1)^{\tilde{p}}(s^p + s^{-p}). \quad (12)$$

For an integer k , denote

$$\{k\}_s = \text{sign}(k) \cdot \sum_{j=1}^{|k|} s^{2j-1-|k|} = \begin{cases} (s^k - s^{-k})/(s - s^{-1}), & s \notin \{\pm 1\}, \\ ks^{k-1}, & s \in \{\pm 1\}; \end{cases} \quad (13)$$

it can be written as a polynomial in $s + s^{-1}$. Noticing

$$A(s^k) = \{1 - k\}_s \cdot X + \{k\}_s \cdot Y,$$

we obtain

$$\rho^{\text{ne}} = (-1)^{\tilde{p}}(\{1 + p\}_s \cdot X + \{-p\}_s \cdot Y), \quad (14)$$

$$\rho^{\text{sw}} = (-1)^{\tilde{q}}(\{1 - q\}_s \cdot X + \{q\}_s \cdot Y), \quad (15)$$

$$\rho^{\text{se}} = (-1)^{\tilde{p} + \tilde{q}}(\{1 + p - q\}_s \cdot X + \{q - p\}_s \cdot Y), \quad (16)$$

and when $\{p\}_s \neq 0$,

$$(\rho^{\text{sw}}, \rho^{\text{se}}) = -(\rho^{\text{nw}}, \rho^{\text{ne}}) \cdot \frac{(-1)^{\tilde{q}-1}}{\{p\}_s} \begin{pmatrix} \{p + q\}_s & (-1)^{\tilde{p}}\{q\}_s \\ -(-1)^{\tilde{p}}\{q\}_s & \{p - q\}_s \end{pmatrix}. \quad (17)$$

As pointed out in the second paragraph of this section, these relations are actually valid for arbitrary X, Y .

Notation 2.2. For $P, X \in \text{SL}(2, \mathbb{C})$, denote PXP^{-1} by $P.X$.

Remark 2.3. For $Z, W \in \text{SL}_0(2, \mathbb{C})$, call (Z, W) *regular* if there exists $P \in \text{SL}(2, \mathbb{C})$ such that $P.Z = A(1)$ and $P.W = A(s)$ with $s + s^{-1} = -\text{tr}(ZW)$. It is easy to see that (Z, W) is non-regular if and only if $-\text{tr}(ZW) = 2a \in \{\pm 2\}$ and $W \neq aZ$, and under this condition, there exists $P \in \text{SL}(2, \mathbb{C})$ such that $P.Z = A(1)$ and $P.W \in \{S_a, S'_a\}$, where

$$S_a = \begin{pmatrix} ai & 1 \\ 0 & -ai \end{pmatrix}, \quad S'_a = \begin{pmatrix} ai & 0 \\ 1 & -ai \end{pmatrix}. \quad (18)$$

According to (14)-(16), the four pairs $(\rho^{\text{nw}}, \rho^{\text{ne}})$, (X, Y) , $(\rho^{\text{nw}}, \rho^{\text{sw}})$ and $(\rho^{\text{sw}}, \rho^{\text{se}})$ are simultaneously regular or not.

3 Representations of Montesinos links

Given a representation ρ , say ρ is *reducible* if all the elements in $\text{Im}(\rho)$ have a common eigenvector; in particular, say ρ is *abelian* if $\text{Im}(\rho)$ is abelian. Call ρ *irreducible* if it is not reducible.

Suppose ρ is a representation of $T = M(p_1/q_1, \dots, p_r/q_r)$. Let ρ_ℓ denote its restriction to $[p_\ell/q_\ell]$, let (X_ℓ, Y_ℓ) denote the generating pair of ρ_ℓ , and assume $-\text{tr}(X_\ell Y_\ell) = s_\ell + s_\ell^{-1}$. Up to conjugacy we may assume $X_1 = A(1)$.

The $-\text{tr}_h(\rho_\ell)$'s have a common value which we denote by $a + a^{-1}$, then for each ℓ , by (12),

$$(-1)^{\tilde{p}_\ell}(s_\ell^{p_\ell} + s_\ell^{-p_\ell}) = a + a^{-1},$$

with $\tilde{p}_\ell/\tilde{q}_\ell$ defined as in (9); switching s_ℓ with s_ℓ^{-1} if necessary, we have

$$(-1)^{\tilde{p}_\ell}s_\ell^{p_\ell} = a. \quad (19)$$

If ρ is reducible and non-abelian, then $(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}})$ is non-regular for at least one of the ℓ 's, which is, by Remark 2.3, equivalent to that the $(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}})$'s are all non-regular, hence $a \in \{\pm 1\}$ and $s_\ell \in \{\pm 1\}$ with $(-1)^{\tilde{p}_\ell}s_\ell^{p_\ell} = a$. By (14), ρ_ℓ^{ne} and Y_ℓ are determined by each other; by (17),

$$(\rho_\ell^{\text{sw}}, \rho_\ell^{\text{se}}) = -(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}}) \cdot (-1)^{\tilde{q}_\ell-1}s_\ell^{q_\ell}B(q_\ell/p_\ell), \quad (20)$$

$$\text{with } B(w) = \begin{pmatrix} 1+w & aw \\ -aw & 1-w \end{pmatrix}. \quad (21)$$

Observing $B(w)B(w') = B(w + w')$, we have

$$\sum_{\ell=1}^r \frac{q_\ell}{p_\ell} = 0 \quad \text{and} \quad \prod_{\ell=1}^r (-1)^{\tilde{q}_\ell-1}s_\ell^{q_\ell} = 1. \quad (22)$$

Conversely, when $a, s_1, \dots, s_r \in \{\pm 1\}$ satisfy $(-1)^{\tilde{p}_\ell}s_\ell^{p_\ell} = a, \ell = 1, \dots, r$ and (22) holds, an arbitrary $Y_1 \neq aA(1)$ gives rise to $\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}}, \rho_\ell^{\text{sw}}, \rho_\ell^{\text{se}}$ and then X_ℓ, Y_ℓ for all ℓ through $(\rho_{\ell+1}^{\text{nw}}, \rho_{\ell+1}^{\text{ne}}) = -(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}})$, (20) and (14). The X_ℓ, Y_ℓ 's combine to define a non-abelian reducible representation of T .

If $a \in \{\pm 1\}$ and $(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{ne}})$ is regular for each ℓ , then $\rho_\ell^{\text{ne}} = -a\rho_\ell^{\text{nw}}$ and $(\rho_\ell^{\text{nw}}, \rho_\ell^{\text{sw}})$ is also regular. Conversely, given X_2, \dots, X_r and s_1, \dots, s_r such that $(-1)^{\tilde{p}_\ell}s_\ell^{p_\ell} = a \in \{\pm 1\}$ and $(X_{r+1} = X_1 \text{ for convention})$

$$(X_\ell, X_{\ell+1}) \text{ is regular and } \text{tr}(X_\ell X_{\ell+1}) = (-1)^{\tilde{q}_\ell}(s_\ell^{q_\ell} + s_\ell^{-q_\ell}) \quad (23)$$

for each ℓ , there is a unique representation ρ of T such that

$$\rho_\ell^{\text{nw}} = X_\ell, \quad \rho_\ell^{\text{ne}} = -aX_\ell, \quad \rho_\ell^{\text{sw}} = -X_{\ell+1}, \quad \rho_\ell^{\text{se}} = aX_{\ell+1};$$

actually, Y_ℓ is determined by $X_\ell, X_{\ell+1}$ and s_ℓ as in (15):

$$(-1)^{\tilde{q}_\ell}(\{1 - q_\ell\}_{s_\ell} \cdot X_\ell + \{q_\ell\}_{s_\ell} \cdot Y_\ell) = -X_{\ell+1}.$$

Remark 3.1. How many such tuples (X_1, \dots, X_r) are there?

For $b \in \mathbb{C}^\times$, let

$$D(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \quad E(b) = \frac{1}{2} \begin{pmatrix} b + b^{-1} & (b - b^{-1})i \\ -(b - b^{-1})i & b + b^{-1} \end{pmatrix}. \quad (24)$$

It is easy to see that $E(b).A(1) = A(b^2)$, and $Z.A(1) = A(1)$ if and only if $Z = D(b)$ for some b .

Take a square root b_ℓ of $(-1)^{\tilde{q}_1-1} s_1^{q_1}$. Since $-\text{tr}(X_1 X_2) = b_1^2 + b_1^{-2}$, there exists λ_1 such that

$$X_2 = D(\lambda_1).A(b_1^2) = (D(\lambda_1)E(b_1)).A(1),$$

then $-\text{tr}(A(1)((D(\lambda_1)E(b_1))^{-1}.X_3)) = -\text{tr}(X_2 X_3) = b_2^2 + b_2^{-2}$ implies

$$X_3 = (D(\lambda_1)E(b_1)D(\lambda_2)E(b_2)).A(1) \quad \text{for some } \lambda_2.$$

In general,

$$X_\ell = (D(\lambda_1)E(b_1) \cdot \dots \cdot D(\lambda_{\ell-1})E(b_{\ell-1})).A(1). \quad (25)$$

Thus $X_{r+1} = X_1$ is equivalent to

$$D(\lambda_1)E(b_1) \cdot \dots \cdot D(\lambda_r)E(b_r) = D(\lambda) \quad \text{for some } \lambda. \quad (26)$$

On the other hand, one can show that any given $Z \in \text{SL}(2, \mathbb{C})$ can be written as $D(c_1)E(b_{r-1})D(c_2)E(b_r)D(c_3)$ for some c_1, c_2, c_3 , hence any (X_2, \dots, X_r) satisfying (23) can be obtained as follows: take $\lambda_1, \dots, \lambda_{r-2}$ arbitrarily, and then take $\lambda_{r-1}, \lambda_r, \lambda$ to fulfill (26).

Nevertheless, the space of $(\lambda_1, \dots, \lambda_r)$ such that the X_ℓ 's given by (25) satisfy (23) is complicated and deserves more efforts to understand.

Remark 3.2. Note that, ρ is abelian if and only if each $s_\ell \in \{\pm 1\}$; in this case $Y_\ell = s_\ell X_\ell$ and $X_{\ell+1} = (-1)^{\tilde{q}_\ell+1} s_\ell^{q_\ell} X_\ell$.

Suppose ρ is irreducible and $a \notin \{\pm 1\}$. Write $a = |a|e^{i\theta}$, then for each ℓ ,

$$s_\ell = |a|^{1/p_\ell} \exp\left(\frac{i}{p_\ell}(\theta + \tilde{p}_\ell \pi + 2n_\ell \pi)\right) \quad \text{for some } n_\ell \in \mathbb{Z}. \quad (27)$$

By (17),

$$(\rho^{\text{sw}}, \rho^{\text{se}}) = -(\rho^{\text{nw}}, \rho^{\text{ne}})C((-1)^{\tilde{q}_\ell-1} s_\ell^{q_\ell}), \quad (28)$$

$$\text{with} \quad C(w) = \frac{1}{a - a^{-1}} \begin{pmatrix} aw - a^{-1}w^{-1} & w - w^{-1} \\ -(w - w^{-1}) & aw^{-1} - a^{-1}w \end{pmatrix}. \quad (29)$$

Observing $C(w)C(w') = C(ww')$, we are lead to

$$\prod_{\ell=1}^r (-1)^{\tilde{q}_\ell - 1} s_\ell^{q_\ell} = 1, \quad (30)$$

which is, by (10), equivalent to

$$|a|^\mu = 1 \quad \text{with} \quad \mu = \sum_{\ell=1}^r \frac{q_\ell}{p_\ell}, \quad (31)$$

$$\mu\theta + \pi \sum_{\ell=1}^r \frac{2n_\ell q_\ell + 1}{p_\ell} = (2n + r)\pi \quad \text{for some } n \in \mathbb{Z}. \quad (32)$$

If $\mu \neq 0$, then $|a| = 1$ (which is equivalent to $-2 < \text{tr}_h(\rho) < 2$), and

$$\theta = \frac{\pi}{\mu} \left(2n + r - \sum_{\ell=1}^r \frac{2n_\ell q_\ell + 1}{p_\ell} \right) \notin \pi\mathbb{Z}. \quad (33)$$

We may assume

$$0 \leq n_\ell < p_\ell, \quad \ell = 1, \dots, r \quad \text{and} \quad 0 \leq n < N(\mu), \quad (34)$$

where $N(\mu)$ is the numerator of μ .

If $\mu = 0$, then a can be arbitrary, and (n_1, \dots, n_r) should satisfy

$$\sum_{\ell=1}^r \frac{2n_\ell q_\ell + 1}{p_\ell} = 2n + r. \quad (35)$$

Theorem 3.3. *Each conjugacy class of tracefree representations of the Montesinos link $M(p_1/q_1, \dots, p_r/q_r)$ contains a unique ρ with $X_1 = A(1)$ and*

- (i) *if ρ is abelian, then it is determined by a unique tuple $(a, s_1, \dots, s_r) \in \{\pm 1\}^{r+1}$ with (19);*
- (ii) *if ρ is reducible but not abelian, then $\mu = 0$ and up to the two choices $Y_1 \in \{S_a, S'_a\}$, ρ is determined by a unique tuple $(a, s_1, \dots, s_r) \in \{\pm 1\}^{r+1}$ satisfying (19) and (30);*
- (iii) *if ρ is irreducible with $-\text{tr}_h(\rho) = a + a^{-1} \in \{\pm 2\}$, then ρ is determined by a and a unique tuple $(s_1, \dots, s_r; X_2, \dots, X_r)$ such that (19) and (23) are satisfied and $s_\ell \notin \{\pm 1\}$ for at least one ℓ ;*

- (iv) if $\mu = 0$ and ρ is irreducible with $-\mathrm{tr}_h(\rho) = a + a^{-1} \notin \{\pm 2\}$, then ρ is determined by a and a unique tuple $(n, n_1, \dots, n_r) \in \mathbb{Z}^{r+1}$ with (34) and (35);
- (v) if $\mu \neq 0$ and ρ is irreducible with $\mathrm{tr}_h(\rho) \notin \{\pm 2\}$, then $-2 < \mathrm{tr}_h(\rho) < 2$ and ρ is determined by a unique tuple $(n, n_1, \dots, n_r) \in \mathbb{Z}^{r+1}$ with (34) and $2n + r - \sum_{\ell=1}^r (2n_\ell q_\ell + 1)/p_\ell$ is not an integral multiple of μ .

Remark 3.4. As pointed out in [6] (see Page 2), the case (ii) never occur if $M(p_1/q_1, \dots, p_r/q_r)$ is a knot.

Remark 3.5. Based on this classifying result, without too much difficulty, one may determine trace-free $\mathrm{SU}(2)$ -representations of a Montesinos link.

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